NPSOR-92-010

NAVAL POSTGRADUATE SCHOOL

Monterey, California

LOWER CONFIDENCE LIMIT EXPRESSIONS FOR P(X>y) AND P(X>Y) UNDER NORMALITY

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December 1991

Prepared for:
Naval Postgraduate School
Monterey, CA

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

This report was prepared for and funded by the Naval Postgraduate School, Monterey, CA.

This report was prepared by:

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY 2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	3. DISTRIBUTION /AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NPSOR-92-010	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Naval Postgraduate School 6b. OFFICE SYMBOL (*If applicable*) OR	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (*City, State, and ZIP Code*) Monterey, CA 93943	7b. ADDRESS (*City, State, and ZIP Code*)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Naval Postgraduate School 8b. OFFICE SYMBOL (*If applicable*)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN, Direct Funding
8c. ADDRESS (*City, State, and ZIP Code*) Monterey, CA 93943	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (*Include Security Classification*)
Lower Confidence Limit Expressions for P(X>y) and P(X>Y) under Normality

12. PERSONAL AUTHOR(S)
W. Max Woods

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (*Year, month day*)	15. PAGE COUNT
Technical Report	FROM ____ TO ____	December, 1991	16

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (*Continue on reverse if necessary and identify by block number*)
FIELD	GROUP	SUB-GROUP	Lower confidence limit expressions for P(X>y) and P(X>Y) under Normality

19. ABSTRACT (*Continue on reverse if necessary and identify by block number*)

Lower confidence limit expressions for P(X>y) and P(X>Y) are provided when both X and Y have Normal probability distributions with unknown means and variances. The cases of equal variances and unequal variances are treated separately. The expressions are approximate but highly accurate as shown in the report.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSICIATION UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL W. Max Woods	22b. TELEPHONE (*Include Area Code*) (408) 646-2743 / 2c. OFFICE SYMBOL OR/Wo

DD Form 1473, JUN 86 *Previous editions are obsolete.* SECURITY CLASSIFICATION OF THIS PAGE
S/N 0102-LF-014-6603 UNCLASSIFIED

LOWER CONFIDENCE LIMIT EXPRESSIONS FOR $P(X>y)$ AND $P(X>Y)$ UNDER NORMALITY

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Closed form expressions of approximate lower confidence limits for $P(X>y)$ and $P(X>Y)$ are presented where X and Y are independent and have normal probability distributions with unknown means and variances. Each of the three expressions requires values of percentile points from the standard t distribution and values of the standard normal cumulative distribution function to compute the lower confidence limit using a data set. The expressions are shown to be quite accurate for sample sizes of ten or larger.

KEY WORDS: Mechanical reliability; interval estimates.

1. INTRODUCTION

Throughout this paper X and Y are independent normally distributed variables with unknown means μ_x and μ_y and unknown variances σ_x^2 and σ_y^2. $\Phi(z)$ is the standard normal CDF, and $t_{\alpha,n}$ is the 100(1−α)th percentile point of the standard t distribution with n degrees of freedom. The symbol $t_n(p)$ denotes a noncentral t variable with n degrees of freedom and noncentrality parameter p. The terms $\bar{X}$ and S_x^2 are the sample mean and sample variance of a random sample of size n on X. The terms $\bar{Y}$ and S_y^2 are the sample mean and sample variance of a random sample of size m on Y.

$S_p^2 = [(n-1)S_x^2 + (m-1)S_y^2] \,/\, (n+m-2)$ and $\bar{x}$, $\bar{y}$, s_x, s_y, s_p denote observed values of the corresponding random variables.

If the random strength, X, of a device must exceed some worst case stress value y, the device reliability is often modeled as $P(X>y)$. If the stress variable, Y, is also random the device reliability is frequently modeled as $P(X>Y)$ which is the average of $P(X>y)$ with respect to the distribution of Y when X and Y are independent. In this paper we present closed expressions for approximate lower confidence limits for $P(X>y)$ and $P(X>Y)$. In the latter case one expression is provided assuming $\sigma_x = \sigma_y$ and a different expression is given assuming $\sigma_x \neq \sigma_y$.

Several papers in the literature present the minimum variance unbiased estimator for $P(X>y)$ when μ_x and σ_x are unknown. See, for example, Lieberman and Resnikoff (5), Folks and others (4), and Barton (1). Tables of exact interval estimates for $P(X>y)$ were developed by Owen and Hua (6) for confidence levels of 90% and 95%. The exact procedure requires a search in tables of the non-central t distribution. Owen and Hua have performed this task and developed more convenient tables. One enters their tables using confidence level $1-\alpha$, sample size n and sample statistic value, $(\bar{x}-y)/s \equiv k$ to obtain the lower confidence limit for P(X>y). The lower limit values obtained using our expression agree with their values to the nearest hundredth decimal point or better for sample sizes of ten or larger and $1 < k \leq 5$. Access to values of $t_{\alpha,m}$ and values of $\Phi(z)$ are required to use our closed form procedure which is also a function of α, n and k.

If $\sigma_x = \sigma_y$ in the two variable case, our lower confidence limit expression for $P(X>Y)$ is analogous to the single variate case. Values of $t_{\alpha,n}$ and $\Phi(z)$ are needed to use this procedure which is a function of sample sizes n_x, n_y and $(\bar{x}-\bar{y}) / S_p$.

Approximate confidence limits for $P(X>Y)$ have been developed by Church and Harris (2) when Y has a standard normal distribution. Downton (3) modified their procedure slightly to obtain more accurate limits and suggests an approximate procedure when the means and variances of both X and Y are unknown.

2. SUMMARY OF RESULTS

For the single variable case we seek a lower $100(1-\alpha)\%$ confidence limit for $P(X>y) \equiv R(y)$. Let $\delta = (\mu_x - y)/\sigma_x$, then $R(y) = \Phi(\delta)$, and a lower confidence limit $R(y)_L$ for $R(y)$ is given by

$$R(y)_L = \Phi(\delta_L) \tag{1}$$

where δ_L is a lower confidence limit for δ.

Owen and Hua (6) used the general confidence interval procedure to find δ_L. Specifically δ_L is the value of δ for which

$$\begin{aligned} 1-\alpha &= P\left[\left(\overline{X}-y\right)/S \le \left(\bar{x}-y\right)/s\right] \\ &= P\left(t_{n-1}\left(\left(\mu_x - y\right)\sqrt{n}\,/\,\sigma_x\right) \le \frac{\left(\bar{x}-y\right)}{s}\sqrt{n}\right) \end{aligned} \tag{2}$$

Letting $k = (\bar{x}-y)/s$, Owen and Hua develop tables of $\Phi(\delta_L)$ for many values of k in $[-3, 6]$, and sample sizes n = 2(1)18, 21(3)30, 40(20)100 and $1-\alpha$ = .90 and .95. The corresponding values of δ_L can easily be obtained from $\Phi^{-1}(\Phi(\delta_L))$.

In Section 3 of this paper an approximate lower confidence limit, δ_L', for δ is shown to be

$$\delta_L' = k - \left\{\frac{1}{n} + \frac{k^2}{2(n-1)}\right\}^{\frac{1}{2}} t_{\alpha,n-1}. \qquad (3)$$

Values of δ_L' were compared with corresponding values of δ_L from Owen and Hua's tables for numerous sets of (n,k) to obtain a more accurate expression, δ_L^*. Let

$$N(\alpha,n) = \begin{cases} 60 & \text{if } \alpha = .05, \quad 10 \le n \le 60 \\ n & \text{if } \alpha = .05, \quad n \ge 60 \\ n & \text{if } \alpha = .10, \quad n \ge 10 \end{cases} \qquad (4)$$

$$\delta_L^* = k - \left\{\frac{1}{n} + \frac{k^2}{2(n-1)}\right\}^{\frac{1}{2}} t_{\alpha,N(\alpha,n)} \qquad (5)$$

The corresponding approximate 100(1–α)% lower confidence limit $R^*(y)_L$ is

$$R^*(y)_L = \Phi\left(\delta_L^*\right). \qquad (6)$$

Table 1 displays values of $R^*(y)_L$ and those given by Owen and Hua, R_L.

For the two variate case, we seek a lower 100(1–α)% confidence limit for $P(X>Y)$. We first assume $\sigma_x = \sigma_Y = \sigma$. Let $R = P(X>Y)$ and $d = (\mu_x - \mu_Y)/\sigma$. then $R = \Phi(d/\sqrt{2})$ and a lower confidence limit, R_L, for R is

$$R_L = \Phi(d_L/\sqrt{2}) \qquad (7)$$

It is shown in Section 3 that a lower 100(1–α)% confidence limit, d_L, for d is the value of d for which

$$1-\alpha = P\left[t\left((\mu_X - \mu_Y)/\sigma\sqrt{(1/n)+(1/m)}\right) \le (\bar{x}-\bar{y})/sp\sqrt{(1/n)+(1/m)}\right] \qquad (8)$$

TABLE 1. APPROXIMATE (R_L^*) AND EXACT (R_L) CONFIDENCE LIMITS FOR P(X>y)

		$\alpha = .10$		$\alpha = .05$	
n	k	R_L^*	R_L	R_L^*	R_L
10	1	.6768	.6816	.6334	.6305
	2	.8890	.8913	.8535	.8519
	3	.9736	.9745	.9560	.9556
	4	.9958	.9960	.9903	.9903
	5	.9996	.9996	.9985	.9985
18	1	.7298	.7303	.6460	.6950
	2	.9260	.9257	.9040	.9035
	3	.9877	.9875	.9800	.9800
	4	.9988	.9988	.9973	.9973
	5	.9999	.9999	.9998	.9998
30	1	.7597	.7594	.7337	.7336
	2	.9431	.9426	.9286	.9285
	3	.9925	.9923	.9885	.9885
	4	.9995	.9994	.9989	.9989
	5	.99998	.99998	.99994	.99994
60	1	.7865	.7861	.7688	.7691
	2	.9562	.9559	.9478	.9480
	3	.9954	.9953	.9936	.9937
	4	.9998	.9998	.9996	.9996
	5	.99999	.99999	.99999	.99999

Alternatively, an approximate lower confidence limit $d'_{L\alpha}$ using a Taylor series expansion of $(\bar{x}-\bar{y})/s_p$ in the manner described in Section 3 is

$$d'_{L\alpha} = K - \left((1/n) + (1/m) + K^2 / 2(n+m-2)\right)^{\frac{1}{2}} t_{\alpha,m+n-2} \tag{9}$$

where $K = (\bar{x}-\bar{y})/s_p$. Yang (8) used Monte Carlo simulations to evaluate the accuracy of $R'_{L\alpha} = \Phi(d'_{L\alpha}/\sqrt{2})$ for α = .20, .10, and .05. One thousand replications of $R'_{L\alpha}$ were generated for each parameter set (σ, μ_x, μ_y, m, n). Values of 1 and 20 were chosen for σ then values of μ_x and μ_y selected so that R = .90, .95 and .99. For each of these six parameter sets, three pairs of sample sizes were chosen for a total of eighteen parameter sets for each value of α. The results of the simulations are given in Table 2. If $R'_{L\alpha}$ is an exact

TABLE 2. ANALYSIS OF APPROXIMATE CONFIDENCE LIMITS FOR $P(X>Y)$: EQUAL VARIANCES CASE

R	n	m	$R'_{L1000(1-\alpha)}$, p $\alpha = .20$	$\alpha = .10$	$\alpha = .05$
.900	8	8	.8989, .801	.8944, .906	.8884, .960
	8	30	.9003, .798	.9050, .888	.8987, .952
	20	30	.9019, .788	.9034, .888	.9015, .947
.950	8	8	.9498, .801	.9467, .909	.9406, .962
	8	30	.9511, .791	.9529, .886	.9489, .951
	20	30	.9516, .785	.9517, .889	.9500, .950
.990	8	8	.9901, .797	.9886, .911	.9865, .962
	8	30	.9906, .783	.9909, .881	.9900, .950
	20	30	.9904, .789	.9904, .895	.9906, .945

procedure, the values in the column labeled $R'_{L1000(1-\alpha)}$ should equal the values of R in the same row. The symbol p denotes the proportion of the

1000 lower confidence limits that covered the value of R. The results were the same for $\sigma = 1$ and $\sigma = 20$.

If we assume $\sigma_x \neq \sigma_y$, let $R = P(X>Y)$ and $r = (\mu_x - \mu_y)/(\sigma_x^2 + \sigma_y^2)^{1/2}$, then $R = \Phi(r)$ and a lower $100(1-\alpha)\%$ confidence limit R_L for R is

$$R_L = \Phi(r_L) \tag{10}$$

where r_L is a lower confidence limit for r. Let $\hat{r}$ be defined by

$$\hat{r} = (\bar{x} - \bar{y}) / \left(s_x^2 + s_y^2\right)^{\frac{1}{2}} \tag{11}$$

The general method cannot be used to find a lower confidence limit for r, because the statistic $\hat{r}$ cannot be modified to an equivalent statistic whose distribution is known with r as the only unknown parameter. This is the same difficulty encountered in the well-known Beherns-Fisher problem associated with finding confidence intervals for $\mu_x - \mu_y$. However an approximate lower confidence limit r_L for r can be found by using a Taylor series expansion of $\hat{r}$ and fitting a t distribution to the distribution of $\hat{r}$ using random degrees of freedom, υ, which is computed from the data. This procedure is developed in Section 3. The expression for $r_{L\alpha}$ is

$$r_{L\alpha} = \hat{r} - \left[\left(s_x^2 / n + s_y^2 / m\right) / \left(s_x^2 + s_y^2\right) + \hat{r}^2\left(s_x^4 / (n-1) + s_y^4 / (m-1)\right) / 2\left(s_x^2 + s_y^2\right)^{3}\right]^{\frac{1}{2}} t_{\alpha,\upsilon} \tag{12}$$

where

$$\upsilon = \left(s_x^2 + s_y^2\right)^2 / \left(s_x^4 / (n-1) + s_y^4 / (m-1)\right) \tag{13}$$

The method used in this procedure to resolve the Beherns-Fisher type of difficulty is similar to that employed by Welch (7). The expression for υ in equation (11) is different from that used by Welch.

Yang (8) has used Monte Carlo simulations to evaluate the accuracy of this procedure given by Equations (8), (9), (10) and (11). The results of his simulations are given in Table 3. The symbol p denotes the proportion of the 1000 randomly generated lower confidence limits that covered the value of R.

TABLE 3. ANALYSIS OF APPROXIMATE CONFIDENCE LIMITS FOR $P(X>Y)$ FOR UNEQUAL VARIANCES CASE

R	σ_X	σ_Y	n	m	$R'_{L1000(1-\alpha)}, p$		
					$\alpha = .20$	$\alpha = .10$	$\alpha = .05$
.900	1.0	2.0	10	20	.9008, .796	.9008, .897	.8959, .956
			25	35	.8996, .803	.8993, .901	.8988, .963
			75	50	.8997, .801	.8990, .907	.8998, .950
	10.0	40.0	10	20	.9019, .791	.9022, .889	.9009, .947
			25	35	.9012, .790	.9005, .897	.8980, .954
			75	50	.9000, .800	.8995, .902	.8992, .952
.950	1.0	2.0	10	20	.9500, .800	.9497, .901	.9461, .955
			25	35	.9502, .798	.9502, .899	.9482, .959
			75	50	.9496, .808	.9493, .903	.9500, .948
	10.0	40.0	10	20	.9513, .784	.9510, .896	.9517, .945
			25	35	.9514, .782	.9494, .902	.9470, .955
			75	50	.9487, .810	.9494, .904	.9500, .950
.990	1.0	2.0	10	20	.9902, .793	.9899, .906	.9888, .955
			25	35	.9903, .789	.9899, .901	.9891, .961
			75	50	.9898, .811	.9899, .905	.9899, .952
	10.0	40.0	10	20	.9906, .777	.9901, .898	.9903, .942
			25	35	.9907, .775	.9904, .893	.9892, .955
			75	50	.9899, .803	.9898, .906	.9900, .949

3. ANALYSIS

For the single variate case we fit a t distribution to the distribution of the statistic $(\bar{x}-y)/s_x$. If $g(\bar{x},s) \equiv (\bar{x}-y)/s_x \equiv k$ is expanded in a Taylor series about (μ_x,σ_x), and the subscript x is dropped, then

$$g(\bar{x},s) \doteq (\mu-y)/s+(\bar{x}-\mu)/\sigma-(s-\sigma)(\mu-y)/s^2.$$

Then

$$E(k) = E[g(\bar{x},s)] \doteq (\mu-y)\left(4-3\sqrt{1-1/(2n-1)}\right)/\sigma$$
$$\doteq (\mu-y)/\sigma \quad \text{if } n \geq 8, \tag{14}$$

and

$$\sigma_k^2 = \mathrm{var}[g(\bar{x},s)] \doteq \mathrm{var}(\bar{x})/\sigma^2+\left((\mu-y)/y^2\right)^2 \mathrm{var}(s)$$
$$\doteq (1/n)+\left((\mu-y)/\sigma\right)^2/2(n-1) \tag{15}$$

Let $\hat{\sigma}_k^2 = (1/n) + k^2/2(n-1)$. It is easily shown that k is a consistent estimator for $(\mu - y)/\sigma \equiv \delta$ and for large n the distribution of k will be approximately normal. Consequently an approximate lower confidence limit for $(\mu - y)/\sigma$ would be $k - \hat{\sigma}_k Z_\alpha$. We choose instead to approximate the distribution of k with a central t distribution with $n - 1$ degrees of freedom and thus obtain the approximate lower $100(1-\alpha)\%$ confidence limit, $\delta_{L\alpha}$, for δ as

$$\delta_{L\alpha} = k-\left((1/n)+k^2/2(n-1)\right)^{\frac{1}{2}} t_{\alpha,n-1} \tag{16}$$

As discussed in Section 2, the tables developed by Owen and Hua can be used to modify the right member of Equation (12) to obtain the more accurate lower confidence limit $\delta^*_{L\alpha}$ given by

$$\delta^*_{L\alpha} = k - \left((1/n) + k^2/2(n-1)\right)^{\frac{1}{2}} t_{\alpha,N(\alpha,n)} \tag{17}$$

where N(α,n) is defined in Equation (4).

In the two variable cases when $\sigma_x = \sigma_y$, we fit a t distribution to the distribution of the statistic $(\bar{x} - \bar{y})/s_p$. Expanding $g(\bar{x}-\bar{y}, s_p) = (\hat{x}-\hat{y})/s_p \equiv K$ in a Taylor series about $(\mu_x - \mu_y, \sigma)$ and collecting terms we get

$$K = g(\bar{x}-\bar{y}, s_p) \doteq \left[(\mu_x - \mu_y)/\sigma\right] + \left[\left(\bar{x} - \bar{y} - (\mu_x - \mu_y)\right)/\sigma\right] - \left(s_p^2 - \sigma^2\right)(\mu_x - \mu_y)/2\sigma^3$$

Then

$$E(K) \doteq \frac{\mu_x - \mu_y}{\sigma} \tag{18}$$

$$\operatorname{var}(K) \doteq ((1/n)) + (1/m)) + (\mu_x - \mu_y)^2 / 2\sigma^2(n+m-2) \tag{19}$$

Let

$$\hat{\sigma}_K = \left((1/n) + (1/m) + K^2/2(n+m-2)\right)^{\frac{1}{2}} \tag{20}$$

Proceeding in a manner similar to the single variate case we obtain the approximate lower confidence limit, $d_{L,\alpha}$, for $d \equiv (\mu_x - \mu_y)/\sigma$ given by

$$d_{L\alpha} = K - \left((1/n) + (1/m) + K^2/2(n+m-2)\right)^{\frac{1}{2}} t_{\alpha,n+m-2} \tag{21}$$

and

$$R_{L\alpha} = \Phi\left(d_{L\alpha}/\sqrt{2}\right) \tag{22}$$

When $\sigma_x \neq \sigma_y$ we fit a t distribution to the distribution of the statistic $\hat{r} \equiv (\bar{x}-\bar{y})/(s_x^2+s_y^2)^{1/2}$. Expanding $\hat{r}$ in a Taylor series about $(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$ and collecting terms we get

$$\hat{r} \doteq (\bar{x}-\bar{y}) / \sqrt{\sigma_x^2+\sigma_y^2} - \left(s_x^2+s_y^2-\left(\sigma_x^2+\sigma_y^2\right)\right)\left(\mu_x-\mu_y\right) / 2\left(\sigma_x^2+\sigma_y^2\right)^{3/2}$$

Consequently

$$E(\hat{r}) \doteq \frac{\mu_x - \mu_y}{\sigma_x^2+\sigma_y^2} \tag{23}$$

$$\begin{aligned} \operatorname{var}(\hat{r}) &\doteq \operatorname{var}(\bar{x}-\bar{y}) / \left(\sigma_x^2+\sigma_y^2\right) + \left(\mu_x-\mu_y\right)^2 \operatorname{var}\left(s_x^2+s_y^2\right) / 4\left(\sigma_x^2+\sigma_y^2\right)^3 \\ &= \left(\left(\sigma_x^2/n+\sigma_y^2/m\right) / \left(\sigma_x^2+\sigma_y^2\right)\right) + \left(\mu_x-\mu_y\right)^2\left(\sigma_x^4/(n-1)+\sigma_y^4/(m-1)\right) / 2\left(\sigma_x^2+\sigma_y^2\right)^3 \end{aligned} \tag{24}$$

Let

$$\hat{\sigma}_{\hat{r}} = \left[\left(s_x^2/n+s_y^2/m\right) / \left(s_x^2+s_y^2\right) + (\bar{x}-\bar{y})^2\left(s_x^4/(n-1)+s_y^4/(m-1\right) / 2\left(s_x^2+s_y^2\right)^3\right]^{\frac{1}{2}} \tag{25}$$

An approximate $100(1-\alpha)\%$ lower confidence limit, r_L, for $(\mu_x - \mu_y)/\left(\sigma_x^2+\sigma_y^2\right)^{1/2} \equiv r$ is

$$r_{L\alpha} = \hat{r} - \hat{\sigma}_{\hat{r}} t_{\alpha,\upsilon} \tag{26}$$

where υ is some appropriate degrees of freedom. Then

$$R_{L\alpha} = \Phi(r_{L\alpha}) \tag{27}$$

The following method for finding an "appropriate" degrees of freedom is used only to develop an expression for the degrees of freedom, υ, that makes equation (23) sufficiently accurate. If a strategy similar to that used in the two previous interval methods is used to find an approximate confidence limit for r, we need to find the approximate degrees of freedom for an equivalent expression for $\hat{r}$ which looks something like a noncentral t distribution. Yang [9] used a different approach to the following procedure, but obtained the same result. The expression

$$\hat{r} = \frac{\bar{x}-\bar{y}}{\sqrt{s_x^2+s_y^2}} = \frac{\dfrac{\bar{x}-\bar{y}-(\mu_x-\mu_y)}{\sqrt{\sigma_x^2+\sigma_y^2}} + \dfrac{(\mu_x-\mu_y)}{\sqrt{\sigma_x 2+\sigma_y^2}}}{\sqrt{\dfrac{s_x^2+s_y^2}{\sigma_x^2+\sigma_y^2}}} \tag{28}$$

suggests, but is not, a noncentral t distribution with noncentrality parameter r. If we were to use a t distribution to fit the distribution of $\hat{r}$, the radicand $(s_x^2 + s_y^2)/(\sigma_x^2 + \sigma_y^2)$ should be of the form χ_c^2/c. Therefore

$$c\left(s_x^2+s_y^2\right)/\left(\sigma_x^2+\sigma_y^2\right)=\chi_c^2 \tag{29}$$

Using the properties $\mathrm{var}\left(\chi_c^2\right)=2c, \mathrm{var}\left(s_x^2\right)=2\sigma_y^4/(n-1), \mathrm{var}\left(s_y^2\right)=2\sigma_y^4/(m-1)$ and taking the variance of both sides of equation (26), we have

$$c^2\left(2\sigma_x^4/(n-1)+2\sigma_y^4/(m-1)\right)/\left(\sigma_x^2+\sigma_y^2\right)^2=2c$$

Solving for c we get

$$c=\left(\sigma_x^2+\sigma_y^2\right)^2/\left(\sigma_x^4/(n-1)+\sigma_y^4/(m-1)\right)$$

Finally we let

$$\upsilon=\hat{c}=\left(s_x^2+s_y^2\right)^2/\left(s_x^4/(n-1)+s_y^4/(m-1)\right) \tag{30}$$

4. CONCLUSIONS

We have derived closed form expressions for approximate lower confidence limits for the reliability of a device when reliability is modeled as $P(X>y)$ or $P(X>Y)$ where X and Y are independent normally distributed variables with unknown means and variances. The expressions have been evaluated for accuracy and demonstrated to be quite accurate when sample sizes are larger than ten. The three confidence limit expressions presented are easy to compute and can be programmed on some existing hand-held calculators. Percentile values, $t_{\alpha,n}$, of the standard t distribution and values of the standardize normal cumulative distribution function, $\Phi(z)$, are required for each expression to compute the lower confidence limit values for given data sets.

In mechanical reliability settings, X usually denotes strength and Y denotes stress. However both reliability models, $P(X>Y)$ and $P(X>y)$, have numerous applications when X and Y denote times to first occurrence of events.

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